

# The $q$ -deformed mKP hierarchy with self-consistent sources, Wronskian solutions and solitons

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**Abstract.** Based on the eigenfunction symmetry constraint of the  $q$ -deformed modified KP hierarchy, a  $q$ -deformed mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs) is constructed. The  $q$ -mKPHSCSs contains two types of  $q$ -deformed mKP equation with self-consistent sources. By combination of the dressing method and the method of variation of constants, a generalized dressing approach is proposed to solve the  $q$ -deformed KP hierarchy with self-consistent sources ( $q$ -KPHSCSs). Using the gauge transformation between the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs, the  $q$ -deformed Wronskian solutions for the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs are obtained. The one-soliton solutions for the  $q$ -deformed KP (mKP) equation with a source are given explicitly.

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## 1. Introduction

In recent years, the  $q$ -deformed integrable systems attracted many interests both in mathematics and in physics (see, e.g., [1-13]). The  $q$ -deformation is performed by using the  $q$ -derivative  $\partial_q$  to take the place of ordinary derivative  $\partial_x$  in the classical systems, where  $q$  is a parameter, and the  $q$ -deformed integrable systems recover the classical ones as  $q \rightarrow 1$ . The  $q$ -deformed integrable systems usually inherit some integrable structures from the classical integrable systems. Take the  $q$ -deformed KP hierarchy as an example, its  $\tau$ -function, bi-Hamiltonian structure and additional symmetries have already been reported (see [7, 8, 9, 13] and the references therein).

Multi-component generalization of an integrable model is a very important subject (see, e.g., [14-16]). For example, the multi-component KP (mcKP) hierarchy given in [14] contains many physically relevant nonlinear integrable systems, such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones. Another type of coupled integrable systems is the soliton equation with self-consistent sources, which has many physical applications and can be obtained by coupling some suitable differential equations to the original soliton equation [17-23]. Recently, a systematical procedure is proposed to construct a KP hierarchy with self-consistent sources and its Lax representation [24, 25]. This idea can be used to the  $q$ -deformed case, i.e., by introducing a new vector field  $\partial_{\tau_k}$  as a linear combination of all vector fields  $\partial_{t_n}$  in the  $q$ -deformed KP hierarchy, then one can get a new Lax type equation which consists of the  $\tau_k$ -flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of  $\partial_{\tau_k}$ -flow and  $\partial_{t_n}$ -flows gives rise to a  $q$ -KP hierarchy with self-consistent sources ( $q$ -KPHSCSs) [26]. The  $q$ -KPHSCSs consists of  $t_n$ -flow,  $\tau_k$ -flow, and  $t_n$ -evolutions of the eigenfunctions and adjoint eigenfunctions. This  $q$ -KPHSCSs contains two types of  $q$ -deformed KP equation with self-consistent sources (1st- $q$ -KPSCS and 2nd- $q$ -KPSCS), and the two kinds of reductions of the  $q$ -KPHSCSs give the  $q$ -deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained  $q$ -deformed KP hierarchy, respectively, which are some (1+1)-dimensional  $q$ -deformed soliton equation with self-consistent sources [26].

The dressing method is an important tool for solving the soliton hierarchy [27]. However, this method can not be applied directly to solve the hierarchy with self-consistent sources. In this paper, with the combination of dressing method and the method of variation of constants, a generalized dressing method for solving the  $q$ -KPHSCSs is proposed. In this way, we can get some solutions of the  $q$ -KPHSCSs in a unified and simple procedure. As a special case, the  $N$ -soliton solutions of the two types of  $q$ -KPSCS's are obtained simultaneously.

To our knowledge, compared to the study on the  $q$ -deformed KP hierarchy, there are less results on the  $q$ -deformed modified KP hierarchy ( $q$ -mKPH). Takasaki studied the  $q$ -mKPH and its quasi-classical limit by considering the  $q$ -analogue of the tau function of the modified KP hierarchy [12]. As another part of this paper, we will present the  $q$ -mKPH explicitly, and then construct an  $q$ -deformed mKP hierarchy with self-

consistent sources ( $q$ -mKPHSCSs) on the base of eigenfunction symmetry constraint. The  $q$ -mKPHSCSs provides a unified way to construct two types of  $q$ -deformed modified KP equation with self-consistent sources (1st- $q$ -mKPSCS and 2nd- $q$ -mKPSCS). Then a gauge transformation between the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs is presented. Since the Wronskian solutions to the  $q$ -KPHSCSs have been obtained by a generalized dressing approach in former part of this paper, the gauge transformation enables us to get the explicit formulation of  $q$ -deformed Wronskian solutions for the  $q$ -mKPHSCSs. It should be noticed that a general setting of “pseudo-differential” operators on regular time scales has been proposed to construct some integrable systems [28, 29], where the  $q$ -differential operator is just a particular case.

This paper will be organized as follows. In Sec. 2, we briefly recall how to construct the  $q$ -KPHSCSs and its Lax representation, and it is shown that the  $q$ -KPHSCSs includes two types of the  $q$ -KPSCS. In Sec. 3, a generalized dressing method for the  $q$ -KPHSCSs will be proposed. In Sec. 4, a  $q$ -deformed mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs) will be constructed, and which includes two types of  $q$ -deformed mKP equation with self-consistent sources. In Sec. 5, the gauge transformation between the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs is established. In Sec. 6, the one-soliton solutions of the  $q$ -deformed KP (mKP) equation with a source are obtained. In the last section, some conclusions and remarks will be given.

## 2. The $q$ -KP hierarchy with self-consistent sources ( $q$ -KPHSCSs)

First, we introduce some useful formula for  $q$ -KP hierarchy. We denote the  $q$ -shift operator and the  $q$ -difference operator by  $\theta$  and  $\partial_q$ , respectively, where  $q$  is a parameter. These operators act on a function  $f(x)$  ( $x \in \mathbf{R}$ ) as

$$\theta(f(x)) = f(qx), \quad \partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)}.$$

In this paper, we use  $P(f)$  to denote an action of a difference operator  $P$  on the function  $f$ , while  $P \circ f = Pf$  means the multiplication of a difference operator  $P$  and a zero- $th$  order difference operator  $f$ , e.g.,  $\partial_q f = (\partial_q(f)) + \theta(f)\partial_q$ .

Let  $\partial_q^{-1}$  denote the formal inverse of  $\partial_q$ . In general, the following  $q$ -deformed Leibnitz rule holds

$$\partial_q^n f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f) \partial_q^{n-k}, \quad n \in \mathbf{Z}, \quad (2.1)$$

where the  $q$ -number and the  $q$ -binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1}, \quad \binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \quad \binom{n}{0}_q = 1.$$

For a  $q$ -pseudo-differential operator ( $q$ -PDO) of the form  $P = \sum_{i=-\infty}^n p_i \partial_q^i$ , we decompose  $P$  into the differential part  $P_+ = \sum_{i=0}^n p_i \partial_q^i$  and the integral part  $P_- = \sum_{i \leq -1} p_i \partial_q^i$ . And the

conjugate operation “ $*$ ” for  $P = \sum_{i=-\infty}^n p_i \partial_q^i$  is defined by  $P^* = \sum_{i=-\infty}^n (\partial_q^*)^i p_i$  with

$$\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q} \partial_{\frac{1}{q}}, \quad (\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta \partial_q^{-1}.$$

The  $q$ -exponent  $e_q(x)$  is defined as

$$e_q(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k\right).$$

Then it is easy to prove that  $\partial_q^k(e_q(xz)) = z^k e_q(xz)$ ,  $k \in \mathbf{Z}$ .

The Lax equation of the  $q$ -KP hierarchy ( $q$ -KPH) is given by (see, e.g., [7])

$$L_{t_n} = [B_n, L] \equiv B_n L - L B_n, \quad (2.2)$$

where  $L = \partial_q + u_0 + u_1 \partial_q^{-1} + u_2 \partial_q^{-2} + \cdots$ ,  $B_n = L_+^n$  stands for the differential part of  $L^n$ .

For any fixed  $k \in \mathbf{N}$ , by introducing a new variable  $\tau_k$  whose vector field is

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^N \sum_{s \geq 0} \zeta_i^{-s-1} \partial_{t_s},$$

where  $\zeta_i$ 's are arbitrary distinct non-zero parameters, then a  $q$ -deformed KP hierarchy with self-consistent sources ( $q$ -KPHSCSs) can be constructed as following [26]

$$L_{\tau_k} = [B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i, L], \quad (2.3a)$$

$$L_{t_n} = [B_n, L], \quad \forall n \neq k, \quad (2.3b)$$

$$\phi_{i,t_n} = B_n(\phi_i), \quad \psi_{i,t_n} = -B_n^*(\psi_i), \quad i = 1, \dots, N. \quad (2.3c)$$

The following proposition is proved in [26].

**Proposition 1** *The commutativity of (2.3a) and (2.3b) under (2.3c) gives rise to the following zero-curvature representation for  $q$ -KPHSCSs (2.3)*

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)_{t_n} + [B_n, B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i] = 0, \quad (2.4a)$$

$$\phi_{i,t_n} = B_n(\phi_i), \quad \psi_{i,t_n} = -B_n^*(\psi_i), \quad i = 1, 2, \dots, N, \quad (2.4b)$$

with the Lax representation given by

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = (B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(\Psi). \quad (2.5)$$

Two kinds of reductions of the  $q$ -KPHSCSs (2.3) are also studied in [26], these reductions give many (1+1)-dimensional  $q$ -deformed soliton equation with self-consistent sources, e.g., the  $q$ -deformed KdV equation with sources, the  $q$ -deformed Boussinesq equation with sources.

For convenience, we write out some operators here

$$B_1 = \partial_q + u_0, \quad B_2 = \partial_q^2 + v_1 \partial_q + v_0, \quad B_3 = \partial_q^3 + s_2 \partial_q^2 + s_1 \partial_q + s_0,$$

$$\phi_i \partial_q^{-1} \psi_i = r_{i1} \partial_q^{-1} + r_{i2} \partial_q^{-2} + r_{i3} \partial_q^{-3} + \dots, \quad i = 1, \dots, N,$$

where

$$v_1 = \theta(u_0) + u_0, \quad v_0 = (\partial_q u_0) + \theta(u_1) + u_0^2 + u_1,$$

$$v_{-1} = (\partial_q u_1) + \theta(u_2) + u_0 u_1 + u_1 \theta^{-1}(u_0) + u_2,$$

$$s_2 = \theta(v_1) + u_0, \quad s_1 = (\partial_q v_1) + \theta(v_0) + u_0 v_1 + u_1,$$

$$s_0 = (\partial_q v_0) + \theta(v_{-1}) + u_0 v_0 + u_1 \theta^{-1}(v_1) + u_2.$$

$$r_{i1} = \phi_i \theta^{-1}(\psi_i), \quad r_{i2} = -\frac{1}{q} \phi_i \theta^{-2}(\partial_q \psi_i), \quad r_{i3} = \frac{1}{q^3} \phi_i \theta^{-3}(\partial_q^2 \psi_i).$$

and  $v_{-1}$  comes from  $L^2 = B_2 + v_{-1} \partial_q^{-1} + v_{-2} \partial_q^{-2} + \dots$ .

Then, one can compute the following commutators

$$[B_2, B_3] = f_2 \partial_q^2 + f_1 \partial_q + f_0, \quad [B_2, \phi_i \partial_q^{-1} \psi_i] = g_{i1} \partial_q + g_{i0} + \dots,$$

$$[B_3, \phi_i \partial_q^{-1} \psi_i] = h_{i2} \partial_q^2 + h_{i1} \partial_q + h_{i0} + \dots, \quad i = 1, \dots, N,$$

where

$$f_2 = \partial_q^2 s_2 + (q+1)\theta(\partial_q s_1) + \theta^2(s_0) + v_1 \partial_q s_2 + v_1 \theta(s_1) + v_0 s_2 - (q^2 + q + 1)\theta(\partial_q^2 v_1) \\ - (q^2 + q + 1)\theta^2(\partial_q v_0) - (q+1)s_2 \theta(\partial_q v_1) - s_2 \theta^2(v_0) - s_1 \theta(v_1) - s_0,$$

$$f_1 = \partial_q^2 s_1 + (q+1)\theta(\partial_q s_0) + v_1 \partial_q s_1 + v_1 \theta(s_0) + v_0 s_1 - \partial_q^3 v_1 - (q^2 + q + 1)\theta(\partial_q^2 v_0) \\ - s_2 \partial_q^2 v_1 - (q+1)s_2 \theta(\partial_q v_0) - s_1 \partial_q v_1 - s_1 \theta(v_0) - s_0 v_1,$$

$$f_0 = \partial_q^2 s_0 + v_1 \partial_q s_0 - \partial_q^3 v_0 - s_2 \partial_q^2 v_0 - s_1 \partial_q v_0,$$

$$g_{i1} = \theta^2(r_{i1}) - r_{i1}, \quad g_{i0} = (q+1)\theta(\partial_q r_{i1}) + \theta^2(r_{i2}) + v_1 \theta(r_{i1}) - r_{i1} \theta^{-1}(v_1) - r_{i2},$$

$$h_{i2} = \theta^3(r_{i1}) - r_{i1}, \quad h_{i1} = (q^2 + q + 1)\theta^2(\partial_q r_{i1}) + \theta^3(r_{i2}) + s_2 \theta^2(r_{i1}) - r_{i1} \theta^{-1}(s_2).$$

$$h_{i0} = (q^2 + q + 1)\theta(\partial_q^2 r_{i1}) + (q^2 + q + 1)\theta^2(\partial_q r_{i2}) + \theta^3(r_{i3}) + (q+1)s_2 \theta(\partial_q r_{i1}) \\ + s_2 \theta^2(r_{i2}) + s_1 \theta(r_{i1}) - r_{i1} \theta^{-1}(s_1) + \frac{1}{q} r_{i1} \theta^{-2}(\partial_q s_2) - r_{i2} \theta^{-2}(s_2) - r_{i3}.$$

Now, we list some examples in the  $q$ -KPHSCSs (2.4) [26].

**Example 1** The first type of  $q$ -deformed KP equation with self-consistent sources (1st- $q$ -KPSCS) is given by (2.4) with  $n = 2$  and  $k = 3$

$$-\frac{\partial s_2}{\partial t_2} + f_2 = 0, \tag{2.6a}$$

$$\frac{\partial v_1}{\partial \tau_3} - \frac{\partial s_1}{\partial t_2} + f_1 + \sum_{i=1}^N g_{i1} = 0, \tag{2.6b}$$

$$\frac{\partial v_0}{\partial \tau_3} - \frac{\partial s_0}{\partial t_2} + f_0 + \sum_{i=1}^N g_{i0} = 0, \tag{2.6c}$$

$$\phi_{i,t_2} = B_2(\phi_i), \quad \psi_{i,t_2} = -B_2^*(\psi_i), \quad i = 1, 2, \dots, N. \tag{2.6d}$$

Let  $q \rightarrow 1$  and  $u_0 \equiv 0$ , then the 1st- $q$ -KPSCS reduces to the first type of KP equation with self-consistent sources [17, 18, 26].

**Example 2** The second type of  $q$ -deformed KP equation with self-consistent sources (2nd- $q$ -KPSCS) is given by (2.4) with  $n = 3$  and  $k = 2$

$$\frac{\partial s_2}{\partial \tau_2} - f_2 + \sum_{i=1}^N h_{i2} = 0, \quad (2.7a)$$

$$\frac{\partial s_1}{\partial \tau_2} - \frac{\partial v_1}{\partial t_3} - f_1 + \sum_{i=1}^N h_{i1} = 0, \quad (2.7b)$$

$$\frac{\partial s_0}{\partial \tau_2} - \frac{\partial v_0}{\partial t_3} - f_0 + \sum_{i=1}^N h_{i0} = 0, \quad (2.7c)$$

$$\phi_{i,t_3} = B_3(\phi_i), \quad \psi_{i,t_3} = -B_3^*(\psi_i), \quad i = 1, 2, \dots, N. \quad (2.7d)$$

Let  $q \rightarrow 1$  and  $u_0 \equiv 0$ , then the 2nd- $q$ -KPSCS reduces to the second type of KP equation with self-consistent sources [17, 26].

### 3. Generalized dressing approach for the $q$ -KPHSCSs

We will first give the dressing approach for the  $q$ -KPH (2.2). Assume that the operator  $L$  of  $q$ -KP hierarchy (2.2) can be written as a dressing form

$$L = S \partial_q S^{-1}, \quad (3.1)$$

with  $S = \partial_q^N + w_1 \partial_q^{N-1} + w_2 \partial_q^{N-2} + \dots + w_N$ .

It is easy to verify that if  $S$  satisfies the Sato equation

$$S_{t_n} = -L_-^n S, \quad (3.2)$$

then  $L$  defined by (3.1) satisfies the  $q$ -KP hierarchy (2.2).

If there are  $N$  linearly independent functions  $h_1, \dots, h_N$  solving  $S(h_i) = 0$ , then  $w_1, \dots, w_N$  are completely determined by solving the linear equations

$$\begin{pmatrix} h_1 & \partial_q(h_1) & \cdots & \partial_q^{N-1}(h_1) \\ h_2 & \partial_q(h_2) & \cdots & \partial_q^{N-1}(h_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_N & \partial_q(h_N) & \cdots & \partial_q^{N-1}(h_N) \end{pmatrix} \begin{pmatrix} w_N \\ w_{N-1} \\ \vdots \\ w_1 \end{pmatrix} = - \begin{pmatrix} \partial_q^N(h_1) \\ \partial_q^N(h_2) \\ \vdots \\ \partial_q^N(h_N) \end{pmatrix}.$$

Then the operator  $S$  can be written as

$$S = \frac{1}{\text{Wr}(h_1, \dots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ \partial_q(h_1) & \partial_q(h_2) & \cdots & \partial_q(h_N) & \partial_q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_q^N(h_1) & \partial_q^N(h_2) & \cdots & \partial_q^N(h_N) & \partial_q^N \end{vmatrix}, \quad (3.3)$$

where

$$\text{Wr}(h_1, \dots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ \partial_q(h_1) & \partial_q(h_2) & \cdots & \partial_q(h_N) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_q^{N-1}(h_1) & \partial_q^{N-1}(h_2) & \cdots & \partial_q^{N-1}(h_N) \end{vmatrix}.$$

**Remark 1** The denominator of  $S$  (3.3) is actually a  $q$ -deformed Wronskian determinant, so we may denote it as  $\text{Wr}_d(h_1, \dots, h_N)$ . The numerator of  $S$  (3.3) is a formal determinant, which is denoted by  $\text{Wr}_d(h_1, \dots, h_N, \partial_q)$ . It is understood as an expansion with respect to its last column, in which all sub-determinants are collected on the left of the difference operator  $\partial_q^j$ .

Then, we have the dressing approach for the  $q$ -KP hierarchy (2.2) as the following.

**Proposition 2** Assume that  $h_i$  satisfies

$$h_{i,t_n} = \partial_q^n(h_i), \quad i = 1, \dots, N, \quad (3.4)$$

and  $S$  and  $L$  are constructed as (3.3) and (3.1) respectively, then  $S$  and  $L$  satisfy the Sato equation (3.2) and the  $q$ -KP hierarchy (2.2).

**Proof** Apply partial derivative  $\partial_{t_n}$  to the equation  $S(h_i) = 0$ , and notice that  $h_i$ 's are linearly independent, then we have  $S_{t_n} + L_-^n S = 0$ . This completes the proof.

Unfortunately, the dressing approach given above can not provide the evolution with respect to the new variable  $\tau_k$ . Now we will generalize the dressing approach to solve the  $q$ -KPHSCSs (2.3) and give exact formulas for  $\phi_i$  and  $\psi_i$ . First, we have the following lemma.

**Lemma 1** For any  $q$ -pseudo-operator  $S$ , if  $S$  satisfies

$$S_{t_n} = -L_-^n S \quad (3.5a)$$

$$S_{\tau_k} = -L_-^k S + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i S. \quad (3.5b)$$

then  $L$  defined by (3.1) satisfies (2.3a) and (2.3b).

**Proof** We only write out the proof for (2.3a).

$$\begin{aligned} L_{\tau_k} &= S_{\tau_k} \partial_q S^{-1} - S \partial_q S^{-1} S_{\tau_k} S^{-1} = (-L_-^k + \sum_i \phi_i \partial_q^{-1} \psi_i) L + L(L_-^k - \sum_i \phi_i \partial_q^{-1} \psi_i) \\ &= [-L_-^k + \sum_i \phi_i \partial_q^{-1} \psi_i, L] = [B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i, L]. \end{aligned}$$

This dressing operator  $S$  can be constructed as following. Let  $f_i$  and  $g_i$  satisfy

$$f_{i,t_n} = \partial_q^n(f_i), \quad f_{i,\tau_k} = \partial_q^k(f_i) \quad (3.6a)$$

$$g_{i,t_n} = \partial_q^n(g_i), \quad g_{i,\tau_k} = \partial_q^k(g_i), \quad i = 1, \dots, N. \quad (3.6b)$$

And let  $h_i$  be the linear combination of  $f_i$  and  $g_i$  as

$$h_i = f_i + \alpha_i(\tau_k) g_i, \quad i = 1, \dots, N, \quad (3.7)$$

with the coefficient  $\alpha_i$  being a differentiable function of  $\tau_k$ . Suppose  $h_1, \dots, h_N$  are linearly independent. Then clearly  $S$  defined by (3.3) and (3.7) still satisfy (3.2) according to Proposition 2. To claim that  $S$  satisfy (3.5b), we present

$$\phi_i = -\dot{\alpha}_i S(g_i), \quad \psi_i = (-1)^{N-i} \theta \left( \frac{\text{Wr}_d(h_1, \dots, \hat{h}_i, \dots, h_N)}{\text{Wr}_d(h_1, \dots, h_N)} \right), \quad (3.8)$$

where the hat  $\hat{\phantom{x}}$  means to rule out this term from the  $q$ -deformed Wronskian determinant, and  $\dot{\alpha}_i = \frac{d\alpha_i}{d\tau_k}$ . Then we have the generalized dressing approach for the  $q$ -KPHSCSs (2.3) as the following proposition.

**Proposition 3** *Let  $S$  be defined by (3.3) and (3.7),  $L = S\partial_q S^{-1}$ ,  $\phi_i$  and  $\psi_i$  be given by (3.8), then  $S$  satisfies (3.5) and  $L$ ,  $\phi_i$ ,  $\psi_i$  satisfy the  $q$ -KPHSCSs (2.3).*

To prove Proposition 3, we need several lemmas. The first one is the  $q$ -deformed version of Oevel and Strampp's lemma [30].

**Lemma 2** (Oevel and Strampp [30]) *Under the condition of Proposition 3, we have*

$$S^{-1} = \sum_{i=1}^N h_i \partial_q^{-1} \psi_i.$$

**Lemma 3** *Under the condition of Proposition 3, we have  $\partial_q^{-1} S^*(\psi_i) = 0$ , for  $i = 1, \dots, N$ .*

**Proof** It can be proved that

$$(\partial_q^{-1} \psi_i S)_- = \partial_q^{-1} S^*(\psi_i). \quad (3.9)$$

Using Lemma 2, we have

$$\begin{aligned} 0 &= (\partial_q^j S^{-1} \circ S)_- = (\partial_q^j \sum_{i=1}^N h_i \partial_q^{-1} \psi_i S)_- = (\sum_{i=1}^N \partial_q^j (h_i) \partial_q^{-1} \psi_i S)_- \\ &= \sum_{i=1}^N \partial_q^j (h_i) \partial_q^{-1} S^*(\psi_i), \quad j = 0, 1, 2, \dots \end{aligned}$$

Solving the equations with respect to  $\partial_q^{-1} S^*(\psi_i)$ , we get Lemma 3.

**Lemma 4** *Under the condition of Proposition 3, the operator  $\partial_q^{-1} \psi_i S$  is a non-negative difference operator and*

$$(\partial_q^{-1} \psi_i S)(h_j) = \delta_{ij}, \quad 1 \leq i, j \leq N. \quad (3.10)$$

**Proof** Lemma 3 and (3.9) implies that  $\partial_q^{-1} \phi_i S$  is a non-negative difference operator.

We define functions  $c_{ij} = (\partial_q^{-1} \psi_i S)(h_j)$ , then  $\partial_q(c_{ij}) = \psi_i S(h_j) = 0$ , which means  $c_{ij}$  does not depend on  $x$  in the sense of  $q$ -deformed version. From Lemma 2, we find that

$$\begin{aligned} \sum_{i=1}^N \partial_q^k (h_i) c_{ij} &= \sum_{i=1}^N \partial_q^k (h_i c_{ij}) = \partial_q^k (\sum_{i=1}^N h_i c_{ij}) = \partial_q^k (\sum_i h_i \partial_q^{-1} \psi_i S(h_j)) \\ &= \partial_q^k (S^{-1} S)(h_j) = \partial_q^k (h_j), \end{aligned}$$

since the functions  $h_1, h_2, \dots, h_N$  are linearly independent, we can easily conclude that  $c_{ij} = \delta_{ij}$ .



*Proof of Proposition 3.*

Analogous to the proof of Proposition 2, we can prove (3.5a). For (3.5b), taking  $\partial_{\tau_k}$  to the identity  $S(h_i) = 0$ , we find

$$\begin{aligned} 0 &= (S_{\tau_k})(h_i) + (S\partial_q^k)(h_i) + \dot{\alpha}_i S(g_i) = (S_{\tau_k})(h_i) + (L^k S)(h_i) - \sum_{j=1}^N \phi_j \delta_{ji} \\ &= (S_{\tau_k} + L_-^k S - \sum_{j=1}^N \phi_j \partial_q^{-1} \psi_j S)(h_i). \end{aligned}$$

Obviously,  $S_{\tau_k} + L_-^k S$  is a pure difference operator of degree  $< N$ . and moreover using Lemma 4,  $\sum_{j=1}^N \phi_j \partial_q^{-1} \psi_j S$  is also a pure difference operator of degree  $< N$ . so  $S_{\tau_k} + L_-^k S - \sum_{j=1}^N \phi_j \partial_q^{-1} \psi_j S$  is a pure difference operator of degree  $< N$ . Since the non-negative difference operator acting on  $h_i$  in the last expression has degree  $< N$ , it can not annihilate  $N$  independent functions unless the operator itself vanishes.

Hence (3.5) is proved. Then Lemma 1 leads to (2.3a) and (2.3b).

The proof of the first equation in (2.3c) is the following.

$$\begin{aligned} \phi_{i,t_n} &= -\dot{\alpha}_i (S(g_i))_{t_n} = -\dot{\alpha} (S_{t_n} + S\partial_q^n)(g_i) \\ &= -\dot{\alpha}_i (-L_-^n S + L^n S)(g_i) = -\dot{\alpha} B_n S(g_i) = B_n(\phi_i). \end{aligned}$$

And it remains to prove the second equation in (2.3c). Firstly, we see that

$$\begin{aligned} (S^{-1})_{t_n} &= ((S^{-1})_{t_n})_- = (-S^{-1} S_{t_n} S^{-1})_- = (S^{-1} (L^n - B_n))_- \\ &= (\partial_q^n S^{-1})_- - (S^{-1} B_n)_- = (\partial_q^n \sum_{i=1}^N h_i \partial_q^{-1} \psi_i)_- - (\sum_{i=1}^N h_i \partial_q^{-1} \psi_i B_n)_- \\ &= \sum_{i=1}^N \partial_q^n (h_i) \partial_q^{-1} \psi_i - \sum_{i=1}^N h_i \partial_q^{-1} B_n^*(\psi_i). \end{aligned}$$

On the other hand,  $(S^{-1})_{t_n} = (\sum_{i=1}^N h_i \partial_q^{-1} \psi_i)_{t_n} = \sum_{i=1}^N \partial_q^n (h_i) \partial_q^{-1} \psi_i - \sum_{i=1}^N h_i \partial_q^{-1} \psi_{i,t_n}$ , so we have  $\sum_{i=1}^N h_i \partial_q^{-1} (B_n^*(\psi_i) + \psi_{i,t_n}) = 0$ . Since  $h_i$ ,  $i = 1, \dots, N$  are linearly independent, it is easy to get  $\psi_{i,t_n} = -B_n^*(\psi_i)$ .

Thus, we proved Proposition 3 (the generalized dressing approach for the  $q$ -KPHSCSs (2.3)).

#### 4. The $q$ -mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs)

In this section, we will construct the  $q$ -deformed mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs). The Lax operator  $\tilde{L}$  of  $q$ -mKP hierarchy is defined by

$$\tilde{L} = \tilde{u} \partial_q + \tilde{u}_0 + \tilde{u}_1 \partial_q^{-1} + \tilde{u}_2 \partial_q^{-2} + \dots.$$

And the Lax equation of  $q$ -mKP hierarchy is given by

$$\tilde{L}_{t_n} = [\tilde{B}_n, \tilde{L}], \quad \tilde{B}_n = (\tilde{L}^n)_{\geq 1}. \quad (4.1)$$

The  $\partial_{t_n}$ -flows are commutative with each other, and we can easily deduce the zero-curvature equation

$$\tilde{B}_{n,t_m} - \tilde{B}_{m,t_n} + [\tilde{B}_n, \tilde{B}_m] = 0. \quad (4.2)$$

When  $n = 2$  and  $m = 3$ , we get the  $q$ -mKP equation. If we take  $q \rightarrow 1$  and  $\tilde{u} \equiv 1$ , then the  $q$ -mKP equation will reduce to the mKP equation

$$4v_t - v_{xxx} + 6v^2v_x - 3(D^{-1}v_{yy}) - 6v_x(D^{-1}v_y) = 0,$$

where  $t := t_3$ ,  $y := t_2$ ,  $v := \tilde{u}_0$ .

According to the squared eigenfunction symmetry (see [31, 32] and the references therein), we can construct a  $q$ -mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs) as

$$\tilde{L}_{\tau_k} = [\tilde{B}_k + \sum_{i=1}^N \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q, \tilde{L}], \quad (4.3a)$$

$$\tilde{L}_{t_n} = [\tilde{B}_n, \tilde{L}], \quad \forall n \neq k, \quad (4.3b)$$

$$\tilde{\phi}_{i,t_n} = \tilde{B}_n(\tilde{\phi}_i), \quad (4.3c)$$

$$\tilde{\psi}_{i,t_n} = -(\partial_q \tilde{B}_n \partial_q^{-1})^*(\tilde{\psi}_i), \quad i = 1, \dots, N. \quad (4.3d)$$

Then it is easy to get the zero curvature equation for the  $q$ -mKPHSCSs (4.3)

$$\tilde{B}_{n,\tau_k} - (\tilde{B}_k + \sum_{i=1}^N \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q)_{t_n} + [\tilde{B}_n, \tilde{B}_k + \sum_{i=1}^N \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q] = 0. \quad (4.4)$$

Under the condition (4.3c) and (4.3d), the Lax pair for the  $q$ -mKPHSCSs (4.3) is given by

$$\Psi_{t_n} = \tilde{B}_n(\Psi), \quad \Psi_{\tau_k} = (\tilde{B}_k + \sum_{i=1}^N \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q)(\Psi).$$

First, for convenience, we write out some operators here

$$\begin{aligned} \tilde{B}_1 &= \tilde{u} \partial_q, & \tilde{B}_2 &= \tilde{v}_2 \partial_q^2 + \tilde{v}_1 \partial_q, & \tilde{B}_3 &= \tilde{s}_3 \partial_q^3 + \tilde{s}_2 \partial_q^2 + \tilde{s}_1 \partial_q, \\ \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q &= \tilde{r}_{i0} + \tilde{r}_{i1} \partial_q^{-1} + \tilde{r}_{i2} \partial_q^{-2} + \dots, & i &= 1, \dots, N, \end{aligned}$$

where

$$\begin{aligned} \tilde{v}_2 &= \tilde{u} \theta(\tilde{u}), & \tilde{v}_1 &= \tilde{u}(\theta(\tilde{u}_0) + \tilde{u}_0 + \partial_q(\tilde{u})), \\ \tilde{v}_0 &= \tilde{u}_1 \theta^{-1}(\tilde{u}) + \tilde{u}_0^2 + \tilde{u} \theta(\tilde{u}_1) + \tilde{u} \partial_q(\tilde{u}_0), \\ \tilde{s}_3 &= \tilde{u} \theta(\tilde{v}_2), & \tilde{s}_2 &= \tilde{u} \partial_q(\tilde{v}_2) + \tilde{u} \theta(\tilde{v}_1) + \tilde{u}_0 \tilde{v}_2, \\ \tilde{s}_1 &= \tilde{u} \partial_q(\tilde{v}_1) + \tilde{u} \theta(\tilde{v}_0) + \tilde{u}_0 \tilde{v}_1 + \tilde{u}_1 \theta^{-1}(\tilde{v}_2), \\ \tilde{r}_{i0} &= \tilde{\phi}_i \theta^{-1}(\tilde{\psi}_i), & \tilde{r}_{i1} &= -\frac{1}{q} \tilde{\phi}_i \theta^{-2}(\partial_q \tilde{\psi}_i), & \tilde{r}_{i2} &= \frac{1}{q^3} \tilde{\phi}_i \theta^{-3}(\partial_q^2 \tilde{\psi}_i), \end{aligned}$$

and  $\tilde{v}_0$  comes from  $\tilde{L}^2 = \tilde{B}_2 + \tilde{v}_0 + \tilde{v}_{-1} \partial_q^{-1} + \dots$ .

Then, one can compute the following commutators

$$\begin{aligned} [\tilde{B}_2, \tilde{B}_3] &= \tilde{f}_3 \partial_q^3 + \tilde{f}_2 \partial_q^2 + \tilde{f}_1 \partial_q, & [\tilde{B}_2, \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q] &= \tilde{g}_{i2} \partial_q^2 + \tilde{g}_{i1} \partial_q + \dots, \\ [\tilde{B}_3, \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q] &= \tilde{h}_{i3} \partial_q^3 + \tilde{h}_{i2} \partial_q^2 + \tilde{h}_{i1} \partial_q + \dots, & i &= 1, \dots, N. \end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_3 &= \tilde{v}_2 \partial_q^2(\tilde{s}_3) + (q+1)\tilde{v}_2 \theta(\partial_q(\tilde{s}_2)) + \tilde{v}_2 \theta^2(\tilde{s}_1) + \tilde{v}_1 \partial_q(\tilde{s}_3) + \tilde{v}_1 \theta(\tilde{s}_2) - (q^2 + q + 1)\tilde{s}_3 \theta(\partial_q^2(\tilde{v}_2)) \\
&\quad - (q^2 + q + 1)\tilde{s}_3 \theta^2(\partial_q(\tilde{v}_1)) - (q+1)\tilde{s}_2 \theta(\partial_q(\tilde{v}_2)) - \tilde{s}_2 \theta^2(\tilde{v}_1) - \tilde{s}_1 \theta(\tilde{v}_2), \\
\tilde{f}_2 &= \tilde{v}_2 \partial_q^2(\tilde{s}_2) + (q+1)\tilde{v}_2 \theta(\partial_q(\tilde{s}_1)) + \tilde{v}_1 \partial_q(\tilde{s}_2) + \tilde{v}_1 \theta(\tilde{s}_1) - \tilde{s}_3 \partial_q^3(\tilde{v}_2) \\
&\quad - (q^2 + q + 1)\tilde{s}_3 \theta(\partial_q^2(\tilde{v}_1)) - \tilde{s}_2 \partial_q^2(\tilde{v}_2) - (q+1)\tilde{s}_2 \theta(\partial_q(\tilde{v}_1)) - \tilde{s}_1 \partial_q(\tilde{v}_2) - \tilde{s}_1 \theta(\tilde{v}_1), \\
\tilde{f}_1 &= \tilde{v}_2 \partial_q^2(\tilde{s}_1) + \tilde{v}_1 \partial_q(\tilde{s}_1) - \tilde{s}_3 \partial_q^3(\tilde{v}_1) - \tilde{s}_2 \partial_q^2(\tilde{v}_1) - \tilde{s}_1 \partial_q(\tilde{v}_1), \\
\tilde{g}_{i2} &= \tilde{v}_2[\theta^2(\tilde{r}_{i0}) - \tilde{r}_{i0}], \\
\tilde{g}_{i1} &= (q+1)\tilde{v}_2 \theta(\partial_q(\tilde{r}_{i0})) + \tilde{v}_2 \theta^2(\tilde{r}_{i1}) + \tilde{v}_1 \theta(\tilde{r}_{i0}) - \tilde{r}_{i0} \tilde{v}_1 - \tilde{r}_{i1} \theta^{-1}(\tilde{v}_2), \\
\tilde{h}_{i3} &= \tilde{s}_3[\theta^3(\tilde{r}_{i0}) - \tilde{r}_{i0}], \\
\tilde{h}_{i2} &= (q^2 + q + 1)\tilde{s}_3 \theta^2(\partial_q(\tilde{r}_{i0})) + \tilde{s}_3 \theta^3(\tilde{r}_{i1}) + \tilde{s}_2 \theta^2(\tilde{r}_{i0}) - \tilde{r}_{i0} \tilde{s}_2 - \tilde{r}_{i1} \theta^{-1}(\tilde{s}_3), \\
\tilde{h}_{i1} &= (q^2 + q + 1)\tilde{s}_3 \theta(\partial_q^2(\tilde{r}_{i0})) + (q^2 + q + 1)\tilde{s}_3 \theta^2(\partial_q(\tilde{r}_{i1})) + \tilde{s}_3 \theta^3(\tilde{r}_{i2}) + (q+1)\tilde{s}_2 \theta(\partial_q(\tilde{r}_{i0})) \\
&\quad + \tilde{s}_2 \theta^2(\tilde{r}_{i1}) + \tilde{s}_1 \theta(\tilde{r}_{i0}) - \tilde{r}_{i0} \tilde{s}_1 + \frac{1}{q} \tilde{r}_{i1} \theta^{-2}(\partial_q(\tilde{s}_3)) - \tilde{r}_{i1} \theta^{-1}(\tilde{s}_2) - \tilde{r}_{i2} \theta^{-2}(\tilde{s}_3).
\end{aligned}$$

Now, we can list the two types of  $q$ -mKP equations with self-consistent source.

**Example 3** When  $n = 2$  and  $k = 3$ , the  $q$ -mKPHSCSs (4.3) gives the first type of  $q$ -mKP equation with self-consistent sources (1st- $q$ -mKPSCS)

$$-\frac{\partial \tilde{s}_3}{\partial t_2} + \tilde{f}_3 = 0, \quad (4.5a)$$

$$\frac{\partial \tilde{v}_2}{\partial \tau_3} - \frac{\partial \tilde{s}_2}{\partial t_2} + \tilde{f}_2 + \sum_{i=1}^N \tilde{g}_{i2} = 0, \quad (4.5b)$$

$$\frac{\partial \tilde{v}_1}{\partial \tau_3} - \frac{\partial \tilde{s}_1}{\partial t_2} + \tilde{f}_1 + \sum_{i=1}^N \tilde{g}_{i1} = 0, \quad (4.5c)$$

$$\tilde{\phi}_{i,t_2} = \tilde{B}_2(\tilde{\phi}_i), \quad \tilde{\psi}_{i,t_2} = -(\partial_q \tilde{B}_2 \partial_q^{-1})^*(\tilde{\psi}_i), \quad i = 1, 2, \dots, N. \quad (4.5d)$$

Let  $q \rightarrow 1$  and  $u \equiv 1$ , then the first type of  $q$ -mKP equation with self-consistent source (4.5) reduces to the first type of mKP equation with self-consistent sources which reads as

$$4\tilde{u}_{0,t} - \tilde{u}_{0,xxx} + 6\tilde{u}_0^2 \tilde{u}_{0,x} - 3D^{-1} \tilde{u}_{0,yy} - 6\tilde{u}_{0,x} D^{-1} \tilde{u}_{0,y} + 4 \sum_{i=1}^N (\tilde{\phi}_i \tilde{\psi}_i)_x = 0,$$

$$\begin{aligned}
\tilde{\phi}_{i,y} &= \tilde{\phi}_{i,xx} + 2\tilde{u}_0 \tilde{\phi}_{i,x}, \\
\tilde{\psi}_{i,y} &= -\tilde{\psi}_{i,xx} + 2\tilde{u}_0 \tilde{\psi}_{i,x}, \quad i = 1, \dots, N,
\end{aligned}$$

where  $t := \tau_3, y := t_2$ .

**Example 4** When  $n = 3$  and  $k = 2$ , the  $q$ -mKPHSCSs (4.3) gives the second type of  $q$ -mKP equation with self-consistent source (2nd- $q$ -mKPSCS)

$$\frac{\partial \tilde{s}_3}{\partial \tau_2} - \tilde{f}_3 + \sum_{i=1}^N \tilde{h}_{i3} = 0, \quad (4.7a)$$

$$\frac{\partial \tilde{s}_2}{\partial \tau_2} - \frac{\partial \tilde{v}_2}{\partial t_3} - \tilde{f}_2 + \sum_{i=1}^N \tilde{h}_{i2} = 0, \quad (4.7b)$$

$$\frac{\partial \tilde{s}_1}{\partial \tau_2} - \frac{\partial \tilde{v}_1}{\partial t_3} - \tilde{f}_1 + \sum_{i=1}^N \tilde{h}_{i1} = 0, \quad (4.7c)$$

$$\tilde{\phi}_{i,t_3} = \tilde{B}_3(\tilde{\phi}_i), \quad \tilde{\psi}_{i,t_3} = -(\partial_q \tilde{B}_3 \partial_q^{-1})^*(\tilde{\psi}_i), \quad i = 1, \dots, N. \quad (4.7d)$$

Let  $q \rightarrow 1$  and  $u \equiv 1$ , then the second type of  $q$ -mKP equation with self-consistent source (4.7) reduces to the second type of mKP equation with self-consistent sources which reads as

$$\begin{aligned} 4\tilde{u}_{0,t} - \tilde{u}_{0,xxx} + 6\tilde{u}_0^2\tilde{u}_{0,x} - 3D^{-1}\tilde{u}_{0,yy} - 6\tilde{u}_{0,x}D^{-1}\tilde{u}_{0,y} \\ + \sum_{i=1}^N [3(\tilde{\phi}_i\tilde{\psi}_{i,xx} - \tilde{\phi}_{i,xx}\tilde{\psi}_i) - 3(\tilde{\phi}_i\tilde{\psi}_i)_y - 6(\tilde{u}_0\tilde{\phi}_i\tilde{\psi}_i)_x] = 0, \\ \tilde{\phi}_{i,t} = \tilde{\phi}_{i,xxx} + 3\tilde{u}_0\tilde{\phi}_{i,xx} + \frac{3}{2}(D^{-1}\tilde{u}_{0,y})\tilde{\phi}_{i,x} + \frac{3}{2}\tilde{u}_{0,x}\tilde{\phi}_{i,x} + \frac{3}{2}\tilde{u}_0^2\tilde{\phi}_{i,x} + \frac{3}{2}\sum_{j=1}^N(\tilde{\phi}_j\tilde{\psi}_j)\tilde{\phi}_{i,x}, \\ \tilde{\psi}_{i,t} = \tilde{\psi}_{i,xxx} - 3\tilde{u}_0\tilde{\psi}_{i,xx} + \frac{3}{2}(D^{-1}\tilde{u}_{0,y})\tilde{\psi}_{i,x} - \frac{3}{2}\tilde{u}_{0,x}\tilde{\psi}_{i,x} + \frac{3}{2}\tilde{u}_0^2\tilde{\psi}_{i,x} + \frac{3}{2}\sum_{j=1}^N(\tilde{\phi}_j\tilde{\psi}_j)\tilde{\psi}_{i,x}, \end{aligned}$$

where  $y := \tau_2, t := t_3$ .

## 5. The gauge transformation between the $q$ -KPHSCSs and the $q$ -mKPHSCSs

In this section, we will give a gauge transformation between the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs.

**Proposition 4** Suppose  $L$ ,  $\phi_i$ 's, and  $\psi_i$ 's satisfy the  $q$ -KPHSCSs (2.3), and  $f$  is a particular eigenfunction for the Lax pair (2.5) of the  $q$ -KPHSCSs, i.e.,

$$f_{t_n} = B_n(f), \quad f_{\tau_k} = (B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(f),$$

then

$$\tilde{L} := f^{-1}Lf, \quad \tilde{\phi}_i := f^{-1}\phi_i, \quad \tilde{\psi}_i := -\theta \partial_q^{-1}(f\psi_i) = (\partial_q^{-1})^*(f\psi_i), \quad (5.1)$$

satisfy the  $q$ -mKPHSCSs (4.3).

**Proof** Since  $f$  is the eigenfunction of the Lax pair (2.5) for the  $q$ -KPHSCSs, then

$$\begin{aligned} \tilde{L}_{t_n} &= (f^{-1}Lf)_{t_n} = -f^{-1}B_n(f)f^{-1}Lf + f^{-1}[B_n, L]f + f^{-1}LB_n(f) \\ &= -f^{-1}B_n(f)\tilde{L} + [f^{-1}B_n f, \tilde{L}] + \tilde{L}f^{-1}B_n(f) = [f^{-1}B_n f - f^{-1}B_n(f), \tilde{L}] = [\tilde{B}_n, \tilde{L}], \end{aligned}$$

here it is used that  $\Delta := f^{-1}B_n f - f^{-1}B_n(f) = f^{-1}[(L^n f)_{\geq 0} - (L^n)_{\geq 0}(f)] = f^{-1}((L^n f)_{\geq 1}) = (f^{-1}L^n f)_{\geq 1} = \tilde{L}_{\geq 1}^n$ , and we denote  $\tilde{L}_{\geq 1}^n$  by  $\tilde{B}_n$ . Moreover, we have

$$\begin{aligned} \tilde{L}_{\tau_k} &= (f^{-1}Lf)_{\tau_k} = -f^{-1}(B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(f)f^{-1}Lf + f^{-1}[B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i, L]f \\ &\quad + f^{-1}L(B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(f) \end{aligned}$$

$$\begin{aligned}
&= [f^{-1}(B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i) f - f^{-1}(B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(f), \tilde{L}] \\
&= [\tilde{B}_k, \tilde{L}] + \sum_{i=1}^N [f^{-1} \phi_i \partial_q^{-1} \psi_i f - f^{-1} \phi_i \partial_q^{-1} \psi_i(f), \tilde{L}] \\
&= [\tilde{B}_k, \tilde{L}] + \sum_{i=1}^N [\tilde{\phi}_i \partial_q^{-1} \circ \partial_q^*(\tilde{\psi}_i) + \tilde{\phi}_i \partial_q^{-1} \circ \partial_q \circ \theta^{-1}(\tilde{\psi}_i), \tilde{L}] \\
&= [\tilde{B}_k, \tilde{L}] + \sum_{i=1}^N [\tilde{\phi}_i \partial_q^{-1} \circ \tilde{\psi}_i \partial_q, \tilde{L}] = [\tilde{B}_k + \sum_{i=1}^N \tilde{\phi}_i \partial_q^{-1} \tilde{\psi}_i \partial_q, \tilde{L}],
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}_{i,t_n} &= -f^{-1} B_n(f) f^{-1} \phi_i + f^{-1} B_n(\phi_i) = -f^{-1} B_n(f) \tilde{\phi}_i + f^{-1} B_n(f \tilde{\phi}_i) \\
&= f^{-1} L_{\geq 0}^n f(\tilde{\phi}_i) - f^{-1} L_{\geq 0}^n(f) \tilde{\phi}_i = (f^{-1} L^n f)_{\geq 0}(\tilde{\phi}_i) - f^{-1} L_{\geq 0}^n(f) \tilde{\phi}_i \\
&= (f^{-1} L^n f)_{\geq 1}(\tilde{\phi}_i) + (f^{-1} L_{\geq 0}^n(f))(\tilde{\phi}_i) - f^{-1} L_{\geq 0}^n(f) \tilde{\phi}_i = \tilde{B}_n(\tilde{\phi}_i),
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}_{i,t_n} &= (\partial_q^{-1})^* [B_n(f) \psi_i - f B_n^*(\psi_i)] = (\partial_q^{-1})^* [B_n(f) f^{-1} \partial_q^*(\tilde{\psi}_i) - f B_n^* f^{-1} \partial_q^*(\tilde{\psi}_i)] \\
&= -(\partial_q^{-1})^* [((f(L^n)^* f^{-1})_{\geq 0} \partial_q^*(\tilde{\psi}_i) - (L^n)_{\geq 0}(f) f^{-1} \partial_q^*(\tilde{\psi}_i))] \\
&= -(\partial_q^{-1})^* [((f^{-1} L^n f)_{\geq 0})^* \partial_q^*(\tilde{\psi}_i) - (L^n)_{\geq 0}(f) f^{-1} \partial_q^*(\tilde{\psi}_i)] \\
&= -(\partial_q^{-1})^* ((f^{-1} L^n f)_{\geq 1})^* \partial_q^*(\tilde{\psi}_i) = -(\partial_q^{-1})^* \tilde{B}_n^* \partial_q^*(\tilde{\psi}_i) = -(\partial_q \tilde{B}_n \partial_q^{-1})^*(\tilde{\psi}_i).
\end{aligned}$$

This completes the proof.

Therefore, if a special eigenfunction  $f$  for the Lax pair (2.5) of the  $q$ -KPHSCSs is given, then we can get a solution of the  $q$ -mKPHSCSs by the gauge transformation (5.1). Here we choose

$$f = S(1) = (-1)^N \frac{\text{Wr}(\partial_q(h_1), \partial_q(h_2), \dots, \partial_q(h_N))}{\text{Wr}(h_1, h_2, \dots, h_N)} \quad (5.2)$$

as the particular eigenfunction for the Lax pair (2.5) of the  $q$ -KPHSCSs, where  $S$  is the dressing operator defined by (3.3) and (3.7). Then the Wronskian solution for the  $q$ -mKPHSCSs is

$$\tilde{L} = f^{-1} L f = \frac{\text{Wr}(h_1, \dots, h_N, \partial_q)}{\text{Wr}(\partial_q(h_1), \dots, \partial_q(h_N))} \partial_q \left[ \frac{\text{Wr}(h_1, \dots, h_N, \partial_q)}{\text{Wr}(\partial_q(h_1), \dots, \partial_q(h_N))} \right]^{-1}, \quad (5.3a)$$

$$\tilde{\phi}_i = f^{-1} \phi_i = -\dot{\alpha}_i \frac{\text{Wr}(h_1, h_2, \dots, h_N, g_i)}{\text{Wr}(\partial_q(h_1), \partial_q(h_2), \dots, \partial_q(h_N))}, \quad (5.3b)$$

$$\tilde{\psi}_i = -\theta \partial_q^{-1}(f \psi_i) = \theta \left( \frac{\text{Wr}(\partial_q(h_1), \dots, \partial_q(\hat{h}_i), \dots, \partial_q(h_N))}{\text{Wr}(h_1, h_2, \dots, h_N)} \right), \quad i = 1, \dots, N. \quad (5.3c)$$

The above expressions for  $\tilde{L}$  and  $\tilde{\phi}_i$ 's can be easily known by straightforward calculation, and the above expressions for  $\tilde{\psi}_i$ 's can be derived as follows. First, we see

$$\sum_{i=1}^N \theta(h_i) \tilde{\psi}_i = \sum_{i=1}^N \theta(-h_i \partial_q^{-1}(\psi_i f)) = \theta((\sum_{i=1}^N -h_i \partial_q^{-1} \psi_i)(f)) = \theta(S^{-1} S(1)) = 1.$$

And moreover we have the following relation (for  $k \geq 1$ ),

$$\begin{aligned}
\sum_{i=1}^N \theta(\partial_q^k(h_i)) \tilde{\psi}_i &= \sum_{i=1}^N \theta[-\partial_q^k(h_i) \cdot \partial_q^{-1}(\psi_i f)] \\
&= \sum_{i=1}^N \theta[-\partial_q(\partial_q^{k-1}(h_i) \cdot \partial_q^{-1}(\psi_i f)) + \theta(\partial_q^{k-1}(h_i) \cdot \psi_i f)] \\
&= \cdots = \sum_{i=1}^N \theta[-\partial_q^k(h_i \partial_q^{-1} \psi_i(f)) + \sum_{j=0}^{k-1} \partial_q^{k-j-1}(\theta(\partial_q^j(h_i)) \psi_i f)] \\
&= \theta[\sum_{j=0}^{k-1} \partial_q^{k-j-1}(\sum_{i=0}^N \theta(\partial_q^j(h_i)) \psi_i f)].
\end{aligned}$$

Notice the definition of  $\psi_i$ 's (3.8) and  $\sum_{i=0}^N \theta(\partial_q^j(h_i)) \psi_i = \delta_{j, N-1}$ ,  $j = 0, 1, \dots, N-1$ , then we have  $\sum_{i=1}^N \theta(\partial_q^k(h_i)) \tilde{\psi}_i = 0$ , for  $k = 1, \dots, N-1$ . Then using the Cramer principle, we can get the exact form of  $\tilde{\psi}_i$ 's (5.3).

## 6. Solutions of the $q$ -KPHSCSs and the $q$ -mKPHSCSs

The generalized dressing approach (Proposition 3) and the gauge transformation (5.1) give us a simple way to construct explicit solutions of the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs. Here we use the first type of  $q$ -KP equation with self-consistent sources (2.6) and the first type of  $q$ -mKP equation with self-consistent sources (4.5) as the examples.

If we choose

$$\begin{aligned}
f_i &:= e_q(\lambda_i x) \exp(\lambda_i^2 t_2 + \lambda_i^3 \tau_3) \equiv e_q(\lambda_i x) e^{\xi_i}, \\
g_i &:= e_q(\mu_i x) \exp(\mu_i^2 t_2 + \mu_i^3 \tau_3) \equiv e_q(\mu_i x) e^{\eta_i}, \\
h_i &:= f_i + \alpha_i(\tau_3) g_i = e_q(\lambda_i x) e^{\xi_i} + \alpha_i(\tau_3) e_q(\mu_i x) e^{\eta_i}, \quad i = 1, \dots, N,
\end{aligned}$$

then the generalized dressing approach (Proposition 3) enables us to get the soliton solutions to the first type of  $q$ -KP equation with sources (2.6).

**Example 5** (One-soliton solution to the 1st- $q$ -KPSCS (2.6)) Let  $N = 1$ , then

$$S = \partial_q + w_0, \quad w_0 = -\frac{\partial_q(h_1)}{h_1}.$$

Notice that  $LS = S\partial_q$ , i.e.,  $(\partial_q + u_0 + u_1\partial_q^{-1} + \cdots)(\partial_q + w_0) = (\partial_q + w_0)\partial_q$ , then the generalized dressing approach (Proposition 3) gives the one-soliton solution to the first type of  $q$ -KP equation with one source ((2.6) with  $N = 1$ )

$$\begin{aligned}
u_0 &= (1 - \theta)(w_0) = (\theta - 1)\left(\frac{\partial_q(h_1)}{h_1}\right) \\
&= \frac{\lambda_1 e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1(\tau_3) \mu_1 e_q(\mu_1 q x) e^{\eta_1}}{e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1(\tau_3) e_q(\mu_1 q x) e^{\eta_1}} - \frac{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1(\tau_3) \mu_1 e_q(\mu_1 x) e^{\eta_1}}{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1(\tau_3) e_q(\mu_1 x) e^{\eta_1}},
\end{aligned}$$

$$\begin{aligned}
u_1 &= -[\partial_q(w_0) + (1 - \theta)(w_0)w_0] = \frac{\partial_q^2 h_1}{h_1} - \left( \frac{\partial_q h_1}{h_1} \right)^2 \\
&= \frac{\lambda_1^2 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1^2 e_q(\mu_1 x) e^{\eta_1}}{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 e_q(\mu_1 x) e^{\eta_1}} - \left( \frac{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 x) e^{\eta_1}}{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 e_q(\mu_1 x) e^{\eta_1}} \right)^2, \\
u_2 &= -u_1 \theta^{-1}(w_0) = u_1 \theta^{-1} \left( \frac{\partial_q(h_1)}{h_1} \right) = u_1 \frac{\lambda_1 e_q(\lambda_1 q^{-1} x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 q^{-1} x) e^{\eta_1}}{e_q(\lambda_1 q^{-1} x) e^{\xi_1} + \alpha_1 e_q(\mu_1 q^{-1} x) e^{\eta_1}}, \\
\phi_1 &= -\dot{\alpha}_1 \frac{h_1 \partial_q(g_1) - \partial_q(h_1) g_1}{h_1} = -\frac{d\alpha_1}{d\tau_3} e_q(\mu_1 x) e^{\eta_1} \left[ \mu_1 - \frac{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 (\tau_3) \mu_1 e_q(\mu_1 x) e^{\eta_1}}{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 (\tau_3) e_q(\mu_1 x) e^{\eta_1}} \right], \\
\psi_1 &= \theta \left( \frac{1}{h_1} \right) = \frac{1}{e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1 (\tau_3) e_q(\mu_1 q x) e^{\eta_1}}.
\end{aligned}$$

**Example 6** (One-soliton solution to the 1st- $q$ -mKPSCS (4.5)) Let  $N = 1$ , then by gauge transformation, the formulae (5.3) gives

$$\tilde{L} = \tilde{u} \partial_q + \tilde{u}_0 + \tilde{u}_1 \partial_q^{-1} + \cdots = (w_1 \partial_q - 1) \partial_q (w_1 \partial_q - 1)^{-1}, \quad w_1 = \frac{h_1}{\partial_q(h_1)},$$

this enables us to get the one-soliton solution to the first type of  $q$ -mKP equation with a source ((4.5) with  $N = 1$ )

$$\begin{aligned}
\tilde{u} &= \frac{w_1}{\theta(w_1)} = \frac{h_1 \theta(\partial_q h_1)}{\partial_q(h_1) \theta(h_1)} \\
&= \frac{(e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 e_q(\mu_1 x) e^{\eta_1})(\lambda_1 e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 q x) e^{\eta_1})}{(\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 (\tau_3) \mu_1 e_q(\mu_1 x) e^{\eta_1})(e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1 (\tau_3) e_q(\mu_1 q x) e^{\eta_1})}, \\
\tilde{u}_0 &= \frac{1}{w_1} [\tilde{u} - 1 - \tilde{u} \partial_q(w_1)] = \frac{\partial_q^2(h_1)}{\partial_q(h_1)} - \frac{\partial_q(h_1)}{h_1} \\
&= \frac{\lambda_1^2 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1^2 e_q(\mu_1 x) e^{\eta_1}}{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 x) e^{\eta_1}} - \frac{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 x) e^{\eta_1}}{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 e_q(\mu_1 x) e^{\eta_1}}, \\
\tilde{u}_1 &= \frac{\tilde{u}_0}{\theta^{-1}(w_1)} = \tilde{u}_0 \frac{\theta^{-1}(\partial_q h_1)}{\theta^{-1}(h_1)} = \tilde{u}_0 \frac{\lambda_1 e_q(\lambda_1 q^{-1} x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 q^{-1} x) e^{\eta_1}}{e_q(\lambda_1 q^{-1} x) e^{\xi_1} + \alpha_1 e_q(\mu_1 q^{-1} x) e^{\eta_1}}, \\
\tilde{\phi}_1 &= -\dot{\alpha}_1 \frac{h_1 \partial_q(g_1) - \partial_q(h_1) g_1}{\partial_q(h_1)} = -\frac{d\alpha_1}{d\tau_3} e_q(\mu_1 x) e^{\eta_1} \left[ \mu_1 \frac{e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 e_q(\mu_1 x) e^{\eta_1}}{\lambda_1 e_q(\lambda_1 x) e^{\xi_1} + \alpha_1 \mu_1 e_q(\mu_1 x) e^{\eta_1}} - 1 \right], \\
\tilde{\psi}_1 &= \theta \left( \frac{1}{h_1} \right) = \frac{1}{e_q(\lambda_1 q x) e^{\xi_1} + \alpha_1 e_q(\mu_1 q x) e^{\eta_1}}.
\end{aligned}$$

## 7. Conclusions

In this paper, we generalized the dressing approach for the  $q$ -KP hierarchy to the  $q$ -KP hierarchy with self-consistent sources ( $q$ -KPHSCSs) by combining the dressing method and the method of variation of constants. The usual dressing method for the  $q$ -KP hierarchy can not provide the evolution to the new  $\tau_k$  variable. By introducing some

varying constants, say  $\alpha(\tau_k)$ , we can obtain the desired evolution to  $\tau_k$ . In this way, we constructed the  $q$ -deformed Wronskian solutions to the  $q$ -KPHSCSs, and got the exact form for the sources  $\phi_i$ 's and  $\psi_i$ 's.

On the base of eigenfunction symmetry constraint, we constructed a  $q$ -mKP hierarchy with self-consistent sources ( $q$ -mKPHSCSs) which contains two series of time variables, say  $t_n$  and  $\tau_k$ . The first and second type of  $q$ -mKP equation with sources ( $q$ -mKPSCS) are obtained as the first two non-trivial equations in the  $q$ -mKPHSCSs. And when  $q \rightarrow 1$  and  $\tilde{u} \equiv 1$ , the  $q$ -mKPHSCSs reduces to the mKP hierarchy with self-consistent sources [25].

A gauge transformation between the  $q$ -KPHSCSs and the  $q$ -mKPHSCSs is established in this paper. By using the gauge transformation, we found the Wronskian solutions for the  $q$ -mKPHSCSs. The one-soliton solutions to the  $q$ -KP equation with a source ((2.6) with  $N = 1$ ) and to the  $q$ -mKP equation with a source ((4.5) with  $N = 1$ ) are given explicitly.

It is interesting to consider if there exist solutions in the  $q$ -deformed case, which are not surviving limit procedure to the classical case. This will be studied in the future.

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